Imperial College
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Lecture 2

Introduction to Systems
(Lathi 1.6-1.8)

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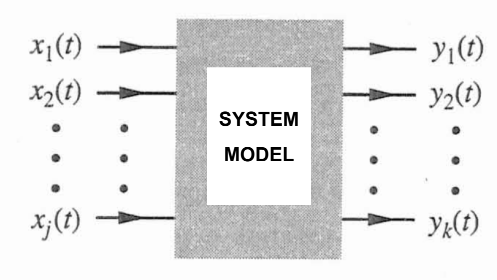
What are Systems?

- Systems are used to **process signals** to **modify** or **extract information**
- Physical system – characterized by their **input-output relationships**
- E.g. electrical systems are characterized by voltage-current relationships for components and the **laws of interconnections** (i.e. Kirchhoff's laws)
- From this, we derive a **mathematical model** of the system
- "**Black box**" model of a system:



Classification of Systems

- Systems may be classified into:
  1. Linear and non-linear systems
  2. Constant parameter and time-varying-parameter systems
  3. Instantaneous (memoryless) and dynamic (with memory) systems
  4. Causal and non-causal systems
  5. Continuous-time and discrete-time systems
  6. Analogue and digital systems
  7. Invertible and noninvertible systems
  8. Stable and unstable systems

Linear Systems (1)

- A **linear system** exhibits the **additivity** property:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad\qquad x_1 + x_2 \longrightarrow y_1 + y_2$$

- It also must satisfy the **homogeneity** or **scaling** property:

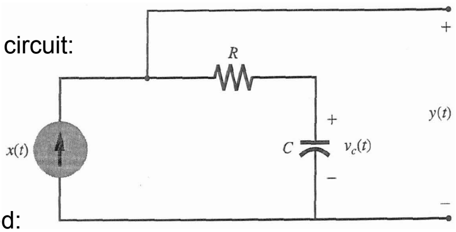
$$x \longrightarrow y \qquad\qquad kx \longrightarrow ky$$

- These can be combined into the property of **superposition**:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad\qquad k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$$

- A non-linear system is one that is NOT linear (i.e. does not obey the principle of superposition)

Linear Systems (2)



- Consider the following simple RC circuit:
- Output *y(t)* relates to *x(t)* by:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

- The second term can be expanded:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{0} x(\tau)\,d\tau + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau$$

$$y(t) = v_C(0) + Rx(t) + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau \qquad t \ge 0$$

- This is a **single-input, single-output** (SISO) system. In general, a system can be multiple-input, multiple-output (**MIMO**).

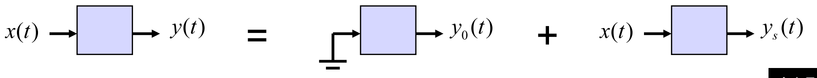
L1.6 (p100)

Linear Systems (3)

- A system's output for t ≥ 0 is result of 2 independent causes:
  1. Initial conditions when t = 0 (**zero-input response**)
  2. Input x(t) for t ≥ 0 (**zero-state response**)
- Decomposition property:

Total response = zero-input response + zero-state response

$$y(t) = \underbrace{v_C(0)}_{\text{zero-input response}} + \underbrace{Rx(t) + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau}_{\text{zero-state response}} \qquad t \ge 0$$



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Linear Systems (4)

- Show that the system described by the equation $\frac{dy}{dt} + 3y(t) = x(t)$ is linear.
- Let $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, then

$$\frac{dy_1}{dt} + 3y_1(t) = x_1(t) \quad \text{and} \quad \frac{dy_2}{dt} + 3y_2(t) = x_2(t)$$

- Multiple 1st equation by k_1, and 2nd equation by k_2, and adding them yields:

$$\frac{d}{dt}[k_1y_1(t) + k_2y_2(t)] + 3[k_1y_1(t) + k_2y_2(t)] = k_1x_1(t) + k_2x_2(t)$$

- This equation is the system equation with

$$x(t) = k_1x_1(t) + k_2x_2(t)$$

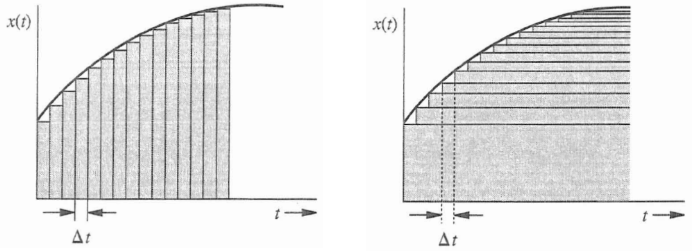
and

$$y(t) = k_1y_1(t) + k_2y_2(t)$$

L1.7-1 p103

Linear Systems (5)

- Almost all systems become **nonlinear** when large enough signals are applied
- Nonlinear systems can be **approximated** by linear systems for **small-signal analysis** – greatly simply the problem
- Once superposition applies, analyse system by decomposition into zero-input and zero-state components
- Equally important, we can represent *x(t)* as a sum of simpler functions (pulse or step)



$$x(t) = a_1x_1(t) + a_2x_2(t) + \cdots + a_mx_m(t)$$

$$y(t) = a_1y_1(t) + a_2y_2(t) + \cdots + a_my_m(t)$$

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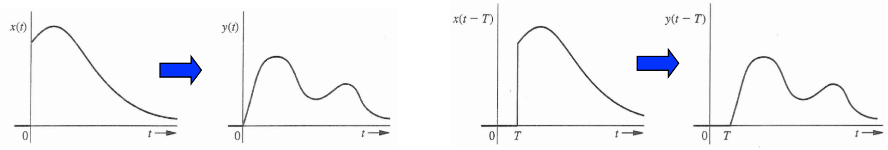
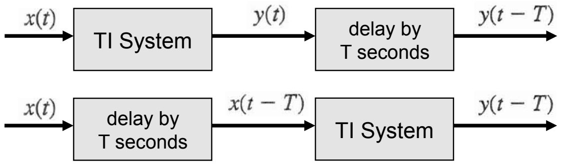
Time-Invariant Systems

- **Time-invariant system** is one whose parameters do not change with time:




$x(t)$ → TI System → $y(t)$ → delay by T seconds → $y(t-T)$

$x(t)$ → delay by T seconds → $x(t-T)$ → TI System → $y(t-T)$

- Linear time-invariant (**LTI**) systems – main concern for this course and the Control course in 2nd year. (Lathi: LTIC = LTI continuous, LTID = LTI discrete)

L1.7-2 p106

Instantaneous and Dynamic Systems

- In general, a system's output at time t **depends** on the entire **past input**. Such a system is a **dynamic** (with memory) **system**
  - Analogous to a state machine in a digital system
- A system whose response at t is completely determined by the input signals over the past T seconds is a **finite-memory** system
  - Analogous to a finite-state machine in a digital system
- Networks containing inductors and capacitors are infinite memory dynamic systems
- If the system's **past history is irrelevant** in determining the response, it is an **instantaneous** or **memoryless** systems
  - Analogous to a combinatorial circuit in a digital system

L1.7-2 p106

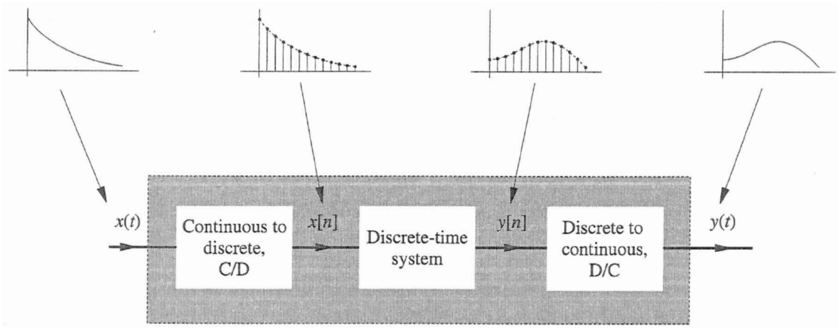
Causal and Noncasual Systems

- **Causal** system – output at t_0 depends only on x(t) *for* $t \le t_0$
- I.e. present output depends only on the past and present inputs, **not on future inputs**
- Any practical **REAL TIME system must be causal**.
- **Noncausal** systems are important because:
  1. Realizable when the independent variable is something other than "time" (e.g. space)
  2. Even for temporal systems, can prerecord the data (non-real time), mimic a non-causal system
  3. Study upper bound on the performance of a causal system

L1.7-4 p108

Continuous-Time and Discrete-Time Systems

- Discrete-time systems process data samples – normally regular sampling of T
- Continuous-time input and output are $x(t)$ and $y(t)$
- Discrete-time input and output samples are $x[nT]$ and $y[nT]$ when n is an integer and $-\infty \le n \le +\infty$



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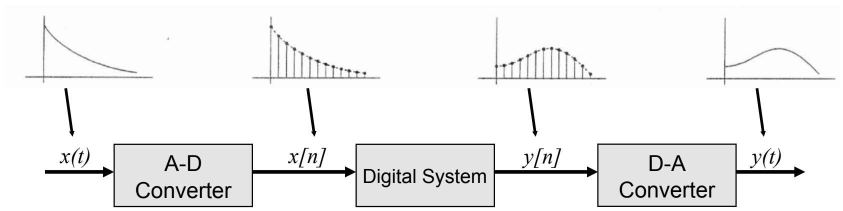
Analogue and Digital Systems

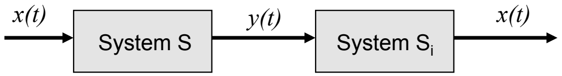
- Previously the samples are discrete in time, but are continuous in amplitude
- Most modern systems are DIGITAL DISCRETE-TIME systems, e.g. internal circuits of the MP3 player



L1.7-5 p111

Invertible and Noninvertible Systems

- Let a system S produces $y(t)$ with input $x(t)$, if there exists another system S_i, which produces $x(t)$ from $y(t)$, then S is invertible
- Essential that there is **one-to-one mapping** between input and output
- For example if S is an amplifier with gain G, it is invertible and S_i is an attenuator with gain 1/G
- Apply S_i following S gives an **identity system** (i.e. input $x(t)$ is not changed)



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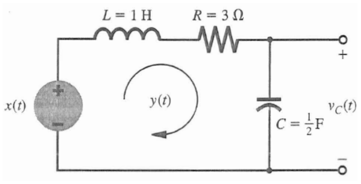
Stable and Unstable Systems

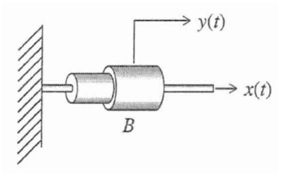
- Externally stable systems: **Bounded input** results in **bounded output** (system is said to be stable in the **BIBO** sense)
- Stability of a system – mostly covered on the Control course

L1.7-8 p112

Linear Differential Systems (1)

- Many systems in electrical and mechanical engineering where input *x(t)* and output loop current *y(t)* are related by **differential equations**
- For example:





$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

$$\frac{dy}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt}$$

$$x(t) = B\dot{y}(t) = B\frac{dy}{dt}$$

L1.8

Linear Differential Systems (2)

- In general, relationship between $x(t)$ and $y(t)$ in a **linear time-invariant** (**LTI**) differential system is given by (where all coefficients a_i and b_i are constants):

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t)$$

$$= b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

- Use compact notation **D** for **operator *d/dt***, i.e $\frac{dy}{dt} \equiv Dy(t)$ and $\frac{d^2 y}{dt^2} \equiv D^2 y(t)$ etc.
- We get: $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$

$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

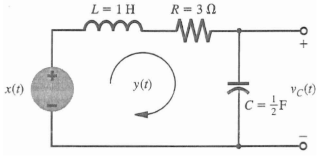
- or $Q(D) y(t) = P(D) x(t)$

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$

$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

L2.1 p151

Linear Differential Systems (3)



- Let us consider this example again:
- The system equation is:

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt}$$

- This can be re-written as:

$$\underbrace{(D^2 + 3D + 2)}_{Q(D)} y(t) = \underbrace{D}_{P(D)} x(t)$$

Also

$$\int_{-\infty}^{t} y(\tau)\, d\tau \equiv \frac{1}{D} y(t)$$

$$\frac{d}{dt}\left[\int_{-\infty}^{t} y(\tau)\, d\tau\right] = y(t)$$

- For this system, $N = 2$, $M = 1$, $a_1 = 3$, $a_2 = 2$, $b_1 = 1$, $b_2 = 0$.
- **For practical systems, M ≤ N**. It can be shown that if M > N, a LTI differential system acts as an (M – N)th-order **differentiator**.
- A differentiator is an unstable system because **bounded input** (e.g. a step input) results in an **unbounded output** (a Dirac impulse δ(t)).

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Relating this lecture to other courses

- Principle of superposition and circuit analysis using differential equations – done in 1st year circuit courses.
- Key conceptual differences: previously bottom-up (from components), more top-down and "black-box" approach.
- Mostly consider mathematical modelling as the key – generalisation applicable not only to circuits, but to other type of systems (financial, mechanical ….)
- Overlap with 2nd year control course, but emphasis is different.
- Equation from last two slides looks similar to transfer function description of system using Laplace Transform, but they are actually different. Here we remain in time domain, and transfer function analysis is in a new domain (s-domain). This will be done later in this course and in the Control course.

Time-domain	s-domain
$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt}$	$(s^2 + 3s + 2)Y(s) = sX(s)$
$\Rightarrow (D^2 + 3D + 2) y(t) = Dx(t)$	$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s}{(s^2 + 3s + 2)}$